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Super theta functions and the Weil representation

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Abstract. We show that the super theta function of Levin is a matrix coefficient of the super Heisenberg group.

1. Introduction

In a well known paper [1], Weil established the foundation of the classical theory of theta functions from a representation-theoretic viewpoint. The classical theta functions were shown to be 'matrix coefficients' of the Shale–Weil–Segal metaplectic oscillator representation. This representation of the extended metaplectic group (i.e. the semi-direct product of the double covering $Mp(n)$ of the symplectic group $Sp(n, \mathbb{R})$ and the $2n$-dimensional Heisenberg group H_n) is the meeting point of several branches of modern mathematics, including number theory, group representation theory and quantum field theory. We relate the super theta function of Levin [2–5] to a super version of the oscillator representation. It is shown in section 4 that there is a geometric picture behind the theory of super theta functions.

2. Rudiments of superalgebras

We start with some generalities on superalgebra. Let Λ be a super commutative associative algebra over $\mathbb{C}$, and $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$ is the decomposition into even and odd elements. A real structure $*$ on Λ is a linear map satisfying the conditions:

(i) $\Lambda_{\bar{0}}^* \subset \Lambda_{\bar{0}}$ and $\Lambda_{\bar{1}}^* \subset \Lambda_{\bar{1}}$

(ii) $(\lambda\alpha)^* = \bar{\lambda}\alpha^*, \forall \lambda \in \mathbb{C}, \alpha \in \Lambda$

(iii) $(\alpha\beta)^* = \alpha^*\beta^*, \forall \alpha, \beta \in \Lambda$

(iv) $\alpha^{**} = (-1)^{\tilde{\alpha}}\alpha, \forall \alpha \in \Lambda$, where $\tilde{\alpha}$ is the degree of an homogeneous α (it is i when $\alpha \in \Lambda_{\bar{i}}$ $(i = 0, 1)$).

The supergroup $GL(n|m, A)$ consists of all invertible super matrices

$$A = \begin{pmatrix} a & \xi \\ \eta & b \end{pmatrix}$$

in block form with even entries of Λ in a, b and odd entries of Λ in η, ξ. The superadjoint is given by

$$A^{\#}=\begin{pmatrix} a^{+} & -\eta^{+} \\ \xi^{+} & b^{+} \end{pmatrix}$$

where $a^{+}=(a_{ji}^{*})$ for $a=(a_{ij})$, and $a_{ij}\in\Lambda$ is the conjugate transpose. Then the super unitary subgroup $U(n|m, \Lambda)=\{A\in GL(n|m, \Lambda)|A^{\#}=A^{-1}\}$ and the orthosymplectic group $OSP(2n|2m, \Lambda)$ is defined as a subgroup of $GL(2n|2m, \Lambda)$ preserving the standard skew-symmetric bilinear form on $\Lambda^{2n|2m}$. $OSP(2n|2m, \Lambda)=\{A\in GL(2n|2m, \Lambda): A^{\#}\mathcal{J}A=\mathcal{J}\}$ where

$$\mathcal{J}=\begin{pmatrix} K & 0 \\ 0 & J \end{pmatrix} \qquad K=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad J=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are of size $n\times n$ and $m\times m$ respectively. So $A\in OSP(2n|2m, \Lambda)$ iff

$$a^{+}Ka-\eta^{+}J\eta=K \qquad a^{+}K\xi=\eta^{+}Jb \qquad \xi^{+}Ka=-b^{+}J\eta \qquad \xi^{+}K\xi+b^{+}Jb=J \tag{1}$$

$$\text{for} \qquad A=\begin{pmatrix} a & \xi \\ \eta & b \end{pmatrix}.$$

Note that the third equation is obtained by taking the adjoint of the second equation, so we indeed have only three equations. Also, $A\in U(n|m, \Lambda)$ iff

$$a^{+}a-\eta^{+}\eta=1 \qquad a^{+}\xi=\eta^{+}b \qquad \xi^{+}\xi+b^{+}b=1 \tag{2}$$

$$A=\begin{pmatrix} a & \xi \\ \eta & b \end{pmatrix}.$$

We make use of the fact that $a^{++}=a$ and $\xi^{++}=-\xi$ for even and odd matrices, respectively. We see further that $OSP(2n|2m, \Lambda)$ contains naturally a $U(n|m, \Lambda)$ as follows: we consider elements of $OSP(2n|2m, \Lambda)$ of the form

$$\begin{pmatrix} a & 0 & 0 & \xi \\ 0 & a & \xi & 0 \\ 0 & \eta & b & 0 \\ \eta & 0 & 0 & b \end{pmatrix}$$

hence a is an $n\times n$ even invertible matrix, ξ is an $n\times m$ odd matrix, etc. Then the defining equation (1) becomes (2). So the subgroup consisting of all these matrices is $U(n|m, \Lambda)$. We see that the homogeneous space $OSP(2n|2m)/U(n|m)$ is a supersymmetric analogue of the Siegel upper half plane when $n=0$ and the compact symmetric space $SO(2n)/U(n)$ when $m=0$.

One can also start with Lie superalgebra over $\mathbb{R}$, say $\mathfrak{R}=\mathfrak{R}_{\bar{0}}\oplus\mathfrak{R}_{\bar{1}}$ with a $\mathbb{Z}/2$ grading. To exponentiate to the corresponding supergroup, we note that in general there are no real points but rather Λ-points, i.e. $\mathfrak{R}_{\Lambda}=\exp\mathfrak{R}_{\Lambda}$ where $\mathfrak{R}_{\Lambda}=(\mathfrak{R}_{\bar{0}}\oplus_{\mathbb{R}}\Lambda_{\bar{0}})\oplus(\mathfrak{R}_{\bar{1}}\oplus_{\mathbb{R}}\Lambda_{\bar{1}})$, where we consider the exponentiation taking place inside the universal enveloping algebra of $\mathfrak{R}_{\Lambda}$. In the following we often drop the dependence of parameters and simply write $OSP(n|m)$, $\mathbb{R}^{n|m}$, etc, and it is understood that the fixed algebra Λ supplies all the parameters.

3. Super Weyl algebra and the super oscillator representation

Let V be a real super vector space of dimension $2n|2m$ with non-degenerate skew-symmetric form $(\ ,\ )$. Let $a_1, \ldots, a_m, a_1^+, \ldots, a_m^+, \alpha_1, \ldots, \alpha_n, \alpha_1^+, \ldots, \alpha_n^+$ be a basis of V such that the matrix of $(\ ,\ )$ with respect to this basis is

$$\mathscr{J}=\begin{pmatrix} 0 & 1 & & 0 \\ -1 & 0 & & \\ & & 0 & 1 \\ 0 & & 1 & 0 \end{pmatrix}$$

i.e. $(a_i, a_j)=(a_i^+, a_j^+)=(\alpha_i, \alpha_j)=(\alpha_i^+, \alpha_j^+)=0$ and $(a_i, a_j^+)=\delta_{ij}=(\alpha_i, \alpha_j^+)$. So $V=V_{\bar 0}\oplus V_{\bar 1}$, $\{a_i, a_i^+\}$ is a basis for $V_{\bar 0}$ and $\{\alpha_i, \alpha_i^+\}$ a basis for $V_{\bar 1}$. We can construct the super Heisenberg algebra $sh(V)$ as a central extension of the Abelian Lie superalgebra V by an even generator c. We have an exact sequence $0\to\mathbb{R}\cdot c\to sh(V)\to V\to 0$, and the Lie bracket is defined by $[u, v]=(u, v)c, \forall u, v\in V$. We also write $sh(V)=sh(2n|2m)$. Let $W(V)$ be the quotient of the universal enveloping algebra $U(h(V))$ by the ideal generated by $c-i$. This is the super Weyl algebra. Therefore $W(V)$ is generated by $a_i, a_i^+, \alpha_i, \alpha_i^+$ if we identify the basis of V as their images in $U(h(V))$ and $W(V)$. The relations are

$$\begin{aligned} &[a_i, a_j]=0=[\alpha_i, \alpha_j]=[a_i^+, a_j^+]=[a_i^+, \alpha_j^+] \\ &[a_i, a_j]=\mathrm{i}\delta_{ij}=[\alpha_i, \alpha_j^+]. \end{aligned} \tag{3}$$

We also write $W(V)=W(2n|2m)$. There is a natural grading in $W(V)$ determined by assigning a_i, a_i^+, α_i and α_i^+ to have degree 1. So $W(V)=\oplus_{i\geq 0} W_i$ where W_i consists of degree i elements. The associated graded algebra gr $W(V)$ is the full (super) symmetric algebra $S(V)$ on V. The quadratic elements $S^2(V)=S^2V_0\oplus(V_0\oplus V_1)\oplus\bigwedge^2 V_1$ inside $W(V)$ act on V as derivations preserving $(\ ,\ )$: that is to say for a linear operator D with degree $\tilde D$ we have

(i) $D(ab)=(Da)b+(-1)^{\tilde a\tilde D}a(Db)$
(ii) $(Da, b)+(-1)^{\tilde a\tilde D}(a, Db)=0, \forall a, b\in V.$

$S^2(V)$ is therefore a Lie superalgebra with even part $S^2(V_0)\oplus\bigwedge^2 V_1$ and odd part $V_0\otimes V_1$. There is an identification of all infinitesimal transformations of V preserving $(\ ,\ )$ with $S^2(V)$, and this subalgebra of End(V) is the orthosymplectic Lie superalgebra $osp(V)=osp(2n|2m)$. In the special cases when $V_0=0$ (respectively $V_1=0$) we recover the Weyl algebra and the symplectic algebra $sp(V)$ on one hand, and the Clifford algebra associated with a symmetric bilinear form on V_1 and the orthogonal algebra.

With respect to the aforementioned basis of V, elements of $osp(2n|2m)$ are super matrices which satisfy the equation $A^{st}+A=0$, where A^{st} is the supertranspose of A [6]. The isomorphism beween $osp(V)$ and $S^2(V)$ can be determined as follows. For all $u\in osp(V)$ let $a(u)$ be the corresponding element in $S^2(V)$. Then for all $v\in V$, $[a(u), v]=u(v)$. The basis of $S^2(V)$ is given by quadratic elements of the form

$$\begin{array}{llll} a_i a_j & \dfrac{a_i^+ a_j + a_j a_i^+}{2} & a_i^+ a_j^+ & \text{for } S^2V_0 \\ a_i\alpha_j, & a_i^+\alpha_j,\ \alpha_i^+ a_j & a_i^+\alpha_j^+ & \text{for } V_0\otimes V_1 \\ \alpha_i\alpha_j & \dfrac{\alpha_i^+\alpha_j - \alpha_j\alpha_i^+}{2} & \alpha_i^+\alpha_j^+ & \text{for } \Lambda^2 V_1. \end{array}$$

We see further that there is a semi-direct sum $osp(V)\oplus V$ of Lie superalgebras: $osp(V)$ acts on V as derivations preserving $(\ ,\ )$. For $u, v\in V$ define $\xi_{u,v}\in osp(V)$ by $\xi_{u,v}(w)=\frac{1}{2}[(u, w)v+(v, w)u]$, then $V_{\bar{0}}$ belongs to the odd part and $V_{\bar{1}}$ belongs to the even part of this enlarged Lie superalgebra. For instance, when $V_{\bar{1}}=0$ we have $sp(V_0)\oplus V_{\bar{0}}\cong osp(1|2n)$. Originally, we can think of the basis a_i, a_j^+ as bosonic oscillators which obey canonical commutator relations. However, when $a_i, a_j^+\in V_{\bar{0}}\subset osp(1|2n)$ we have to impose anti-commutator relations $[a_i a_j^+]_+=a_ia_j^++a_j^+a_i$ for $osp(1|2n)$. This 'parity reversal' phenomenon has been mentioned in Beckers and Cornwell [7] and Günaydin [8]. For the case of $V_{\bar{0}}=0$, we have $o(2m)\oplus V=o(2m+1)$ [9].

Next, we turn to the representation-theoretic significance of the super Weyl algebra. Even though we can view $W(V)$ as a deformation of the (super) symmetric algebra $S(V)$ on V, $W(V)$ is important in the extreme cases when $V_0=0$ or $V_1=0$ to induce the metaplectic (Segal–Shale–Weil) representation for the symplectic algebra (group) and the spinor representation for the orthogonal algebra (group), respectively. Explicitly, the construction is via the action of the Lie algebras on the Fock spaces (bosonic or fermionic). We view the basis elements of V as bosonic or fermionic oscillators where a_i and α_i are creation operators and a_i^+ and α_i^+ are annihilation operators.

Let S be the algera generated by x_i $(1\le i\le m)$, $y_j(1\le j\le n)$ which is symmetric in x_is and antisymmetric in y_js. Therefore S is just the polynomial algebra on x_is tensor the exterior algebra generated by y_js. For $(k, l)=(k_1,\ldots,k_m; l_1,\ldots,l_n)\in\mathbb{Z}^m_{\ge0}\times\{0, 1\}^n$, we define

$$x(k):=\begin{cases} x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m} & (k\in\mathbb{Z}^m_{\ge0})\\ 0 & \text{otherwise}\end{cases}$$

$$y(l):=\begin{cases} y_1^{l_1}\wedge\ldots\wedge y_n^{l_n} & (l\in\{0,1\}^n)\\ 0 & \text{otherwise}\end{cases}. \tag{4}$$

Clearly the set $\{x(k)\cdot y(l)\,|\,(k, l)\in\mathbb{Z}^m_{\ge0}\times\{0, 1\}^n\}$ is an algebraic basis of S as a vector space. $W(V)$ acts on S via

$$a_i(x(k)):=k_ix(k-e_i)$$
$$a_i^+(x(k)):=x(k+e_i)$$
$$\alpha_\mu(y(l)):=(-1)^{l_1+\cdots+l_{\mu-1}}y(l-f_\mu)$$
$$\alpha_\mu^+(y(l)):=(-1)^{l_1+\cdots+l_{\mu-1}}y(l+f_\mu)$$

and

$$x(k):=0 \qquad \text{if some } k_i<0 \tag{5}$$

$\forall k=(l_1,\ldots,l_m)\in\mathbb{Z}^m_{\ge0}$ and $\forall l\in\{0, 1\}^n$ where e_is are elements of $\mathbb{Z}^m_{\ge0}$, f_μs elements of $\{0, 1\}^n$ defined by $e_i=(0,\ldots,1,\ldots,0)$, etc. a_i and a_i^+ act trivially on all $y(l)$s while α_μ and α_μ^+ act trivially on all $x(k)$s. Then we have the following.

Theorem 1. The representation S of $W(V)$ is irreducible.

Proof. Let U be a subrepresentation of S and let $u=\Sigma_{k,l}a_{k,l}x(k)y(l)$ $(a_{k,l}\in\mathbb{C})$ be nonzero in U, then there is a maximal element $k^0=(k_i^0)$ in $\{k\in\mathbb{Z}^m_{\ge0}\,|\,a_{k,l}\ne0\}$ for fixed $l\in\{0, 1\}^n$ according to the lexigraphical order and we can pick the maximal element k^0

among the set $\{0, 1\}^n$, which is finite. Then for this $\boldsymbol{k}^0$, $\boldsymbol{l}^0$

$$a_1^{k_1^0} \ldots a_m^{k_m^0} \alpha_1^{l_1^0} \ldots \alpha_n^{l_n^0} u = a_{k^0, l^0} k_1^0! \ldots k_m^0!\, x(\mathbf{0}) y(\mathbf{0}) \tag{6}$$

so $x(\mathbf{0})y(\mathbf{0}) \in U$. But obviously $S = W \cdot \mathrm{x}(0)\mathrm{y}(0)$ by construction, therefore $U = S$.

S has a natural grading $S = \oplus_{k=0}^{\infty} S^{(k)}$ by assigning x_i and y_i to have degree 1. In particular $S = S_{\mathrm{ev}} \oplus S_{\mathrm{od}}$ where $S_{\mathrm{ev}} := S = \oplus_{k=0}^{\infty} S^{(2k)}$, $S_{\mathrm{od}} := S = \oplus_{k=0}^{\infty} S^{(2k+1)}$.

Since the orthosymplectic algebra $osp(2m|2n)$ embeds as quadratic elements of $W(V)$, S becomes an osp-module. This is called the super oscillator representation of $osp(2m|2n)$ (or Singleton representation in the physics literature [10, 11] because only one set of bosonic and fermionic oscillators are used in the construction).

Theorem 2. The $osp(2m|2n)$ module S_{ev} (respectively S_{od}) is irreducible and is generated by $x(\mathbf{0})y(\mathbf{0})$ (respectively $x(e_n)$).

Also, if we define a Hermitian form $(\ |\ )$ on S such that $(x(\mathbf{0})|x(\mathbf{0})) = 1$ and $(y(\mathbf{0})|y(\mathbf{0})) = 1$ and make S a $*$-representation of $W(V)$ with $a_i^* := a_i^+$ and $\alpha_\mu^* := \alpha_\mu^+$ and an orthogonal basis is given by

$$(x(\boldsymbol{k})y(\boldsymbol{l})\,|\,x(\boldsymbol{k}')y(\boldsymbol{l}')) = \delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\boldsymbol{l}\boldsymbol{l}'} k_1! \ldots k_n! \qquad \forall \boldsymbol{k}, \boldsymbol{k}' \in \mathbb{Z}_{\geq 0}^m \qquad \boldsymbol{l}, \boldsymbol{l}' \in \{0, 1\}^n. \tag{7}$$

Theorem 3. $(\ |\ )$ is the unique Hermitian form satisfying the above properties and $(S, (\ |\ ))$ is then a 'unitary' representation of $W(V)$.

Remark 1. A $*$-algebra is a $\mathbb{C}$-algebra (or superalgebra) with a $\mathbb{C}$ anti-linear algebra automorphism $*: a \mapsto a^*$ such that $*$ is a graded involution. A representation W of $*$-algebra A is a $*$-representation if there exists a Hermitian form $(\ |\ )$ such that $(au|v) = (-1)^{\bar{a}\bar{u}}(u|a^*v) \forall u, v \in W, \forall a \in A$. W is called *unitary* if $(\ |\ )$ is definite.

In the purely even case the representation theory of the Heisenberg algebra and group are fairly simple: there is essentially only one unitary irreducible representation for the Heisenberg group H_{2n}, where the centre acts non-trivially. This essentially means that any two such representations with the same action of the centre are unitarily equivalent. This is the content of the Stone–Von Neumann theorem [12]. Let V be a $2n$-dimensional real vector space. We can associate the Heisenberg algebra $sh(2n|0)$, the Weyl algebra $W(2n|0)$, the symplectic algebra $sp(2n)$ as well as the corresponding groups $H(2n)$, $Sp(2n)$ with V. Among all the equivalent infinite-dimensional representation spaces which give the unique unitary irreducible representation, two are of special interest. The first is the Schwarz space in $L^2(\mathbb{R}^n)$ and this is called the Schrödinger representation $\mu: H(2n) \to \mathrm{Aut}(S(\mathbb{R}^n))$. $H(2n)$ is the simply connected $(2n+1)$-dimensional real Lie group with Lie algebra $sh(2n|0)$. If $(x_1, \ldots, x_n)$ are the standard coordinates on $\mathbb{R}^n$ and $(a_1, \ldots, a_n, a_1^+, \ldots, a_n^+)$ is a basis of V such that $[a_i, a_j^+] = \delta_{ij}\lambda^{-1}c$ where c is the generator of the centre of $H(2n)$ then μ is defined as follows:

$$\begin{aligned}
&\mu(\exp(xa_i))(f)(\boldsymbol{u}) = \exp(2\pi \mathrm{i} u_i) f(\boldsymbol{u}) \\
&\mu(\exp(ya_i^+))(f)(\boldsymbol{u}) = f(\boldsymbol{u} + y) \\
&\mu(\exp(tc)) f(\boldsymbol{u}) = \exp(2\pi \mathrm{i} t\lambda) f(\boldsymbol{u}) \qquad \text{some } \lambda \in \mathbb{C} \text{ for } 1 \leq i \leq n.
\end{aligned} \tag{8}$$

Therefore, infinitesimally we have

$$(\mathrm{d}\mu)(a_i)(f)(u)=2\pi \mathrm{i} u_i f(u)$$
$$(\mathrm{d}\mu)(a_i^+)(f)(u)=\frac{\partial f}{\partial u_i} \tag{9}$$
$$(\mathrm{d}\mu)(c)(f)(u)=2\pi \mathrm{i}\lambda f(u)$$

where $f\in S(\mathbb{R}^n)$. The symplectic group $Sp(2n, \mathbb{R})$ acts as automorphisms of $H(2n)$ and hence from the Stone–von Neumann theorem there exists an intertwining action of $Sp(2n, \mathbb{R})$ on $S(\mathbb{R}^n)$. It is well known that this is not a true representation but rather a projective representation. It can be lifted to a double cover $Mp(2n, \mathbb{R})$ of $Sp(2n, \mathbb{R})$ to become a true representation of this group. This is called the metaplectic or oscillator representation. We describe $Mp_2(\mathbb{R})$ as the set [13]

$$\left\{(A, j^{1/2}(A, \tau)) \text{ with } A\in SL_2(\mathbb{R}), j(A, \tau)=c\tau+d \text{ if } A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \tau\in H\right\}.$$

The lift of any discrete subgroup of $SL_2(\mathbb{R})$ is still discrete in $Mp_2(\mathbb{R})$. The group law in $Mp_2(\mathbb{R})$ is simply given by $(A_1, j^{1/2}(A, \tau))\cdot(A_2, j^{1/2}(A_2, \tau))=(A_1, A_2, j^{1/2}(A_1 A_2, \tau))$, which makes use of the property $j(A_1A_2, \tau)=j(A_1, A_2\cdot\tau)j(A_2, \tau)$. We can view it as a representation of the semi-direct product $Mp(2n)\ltimes H(2n)$. On the other hand, the Fock–Bergman model explicitly shows that in fact $Mp(2n)\ltimes H(2n)$ acts holomorphically.

One would like to extend the above bosonic picture to the supersymmetric case. It is well known that the fermionic picture corresponds to the representation theory of the Jordan algebra, orthogonal algebra and group, and also the Clifford algebra and its spinor representation (see [15], [16]). The super analogue of the Stone–von Neumann theorem should correspond to the following proposition in the even case.

Theorem 4. There exists a unique 'unitary' simple $W(V)$ module.

Proof. Basically this is given by the bosonic Fock space. In the supersymmetric case, if $V=V_{\bar 0}\oplus V_{\bar 1}$ is the decomposition into even and odd parts, the Weyl algebra $W(V)\cong(V_{\bar 0})\otimes W(V_{\bar 1})$ where the canonical isomorphism follows from the super Poincare–Birkoff–Witt theorem. Up to equivalence, we have a unique irreducible $W(V_{\bar 0})$ module, the bosonic Fock space M_0 and a unique irreducible $W(V_{\bar 1})$ module, the finite-dimensional fermionic Fock space, M_1. We take $M:=M_0\otimes M_1$. Then M is an irreducible $W(V)$ module. For the uniqueness part, let N be an irreducible $W(V)$ module and let $N_0:=\mathrm{Hom}_{W(V_1)}(M_1, N)$, then $N_0\otimes_{\mathbb{C}}M_1\cong N$ as a $W(V)$ module. Since N is irreducible as a $W(V)$ module, so must N_0 be as a $W(V_0)$ module, hence $N_0\cong M_0$ and $N\cong M$.

Notice that we need $\dim V=2n|2m$ in order to ensure that the Clifford algebra $W(V_1)$ has only one irreducible module.

Next we consider the extension of the Schrödinger representation to the super Heisenberg algebra $sh(2m|2n)$ and group $sH(2m|2n)$.

The target space for the super Schrödinger representation is $L^2(\mathbb{R}^{n|m}):=L^2(\mathbb{R}^n)\otimes\bigwedge^*(\mathbb{R}^m)^*$ and the smooth vectors are the Schwarz space $S(\mathbb{R}^{n|m}):=S(\mathbb{R}^n)\otimes\bigwedge^*(\mathbb{R}^m)^*$, i.e. the analysis appears solely in the even part. The appropriate Hermitian form is given by

$$\langle f(x,\vartheta), g(x,\vartheta)\rangle = \iint f(x,\vartheta)\overline{g(x,\vartheta)}\,\mathrm{d}x\,\mathrm{d}\vartheta \tag{10}$$

where

$$\int 1\,\mathrm{d}\vartheta = 0 \qquad \int \vartheta_1\ldots\vartheta_m\,\mathrm{d}\vartheta = 1$$

follows the Berezin rule of integration and $(x,\vartheta)=(x_1,\ldots,x_n;\vartheta_1,\ldots,\vartheta_m)$ are the coordinates on $\mathbb{R}^{n|m}$. $\langle\ ,\ \rangle$ is non-degenerate, even and $sh(2n|2m)$ invariant with respect to the super Schrödinger representation.

Infinitesimally $\mathrm{d}\mu$ sends a_j to multiplication by $2\pi\mathrm{i}x_j$, a_j^+ to $\partial/\partial x_j$, α_j to multiplication by $2\pi\mathrm{i}\vartheta_j$ and a_j^+ to $\partial/\partial\vartheta_j$; also, the central element c acts by multiplication by i. We have

$$\langle \mathrm{d}\mu(u)f(x,\vartheta), g\rangle + \langle f, \mathrm{d}\mu(u)g\rangle = 0 \tag{11}$$

for all $u\in sh(2m|2n)$ and $f, g\in L^2(\mathbb{R}^{n|m})$. For the sake of simplicity, we restrict ourselves only to the simplest situation $n=m=1$ without losing generalities. In exponentiating the Lie algebra action $L^2(\mathbb{R}^{1|1})$ to the group level, we obtain

$$\begin{aligned}
&\{\mu(\exp(ka))f\}(x,\vartheta)=\exp(2\pi\mathrm{i}kx)f(x,\vartheta)\\
&\{\mu(\exp(la^+))f\}(x,\vartheta)=f(x+l,\vartheta)\\
&\{\mu(\exp(\sigma\alpha))f\}(x,\vartheta)=\exp(2\pi\mathrm{i}\sigma\vartheta)f(x,\vartheta)\\
&\{\mu(\exp(\rho\alpha^+))f\}(x,\vartheta)=f(x,\vartheta+\rho)
\end{aligned} \tag{12}$$

where (x,ϑ) is the coordinate on $\mathbb{R}^{1|1}$, k, l are even real parameters and σ, ρ are odd real parameters. We call this the super Schrödinger representation of sH.

4. Super theta functions as matrix coefficient

We recall that an even family of supertori is defined as the quotient of $\mathbb{C}^{1|1}$ with the standard superconformal structure ($D:=\partial/\partial\theta+\theta\partial/\partial z$ with respect to global coordinates (z,θ)) by the action of an Abelian subgroup $\equiv\mathbb{Z}^2$ generated by

$$\begin{aligned}
&A:(z,\theta)\mapsto(z+1,\theta)\\
&B:(z,\theta)\mapsto(z+\tau,-\theta) \qquad \text{with } \operatorname{Im}\tau>0
\end{aligned}$$

since this even supertori family

$$\begin{array}{c}\mathbb{C}^{1|1}\times_{\mathbb{Z}^2} H\\ \downarrow\\ H\end{array}$$

gives a split super Riemann surface [16] of genus 1, the minus sign in the last expression specifies the corresponding even spin structure on the underlying torus $X_\tau:=\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$.

The corresponding spinor section is given by

$$\wp_1(z)=\sqrt{\wp(z)-e_1}=\frac{\vartheta_4'(0,\tau)}{\vartheta_1(0,\tau)}\frac{\vartheta_1(z,\tau)}{\vartheta_4(z,\tau)}$$

where $\wp(z)$ is Weierstrass $\wp$-function. The other two even spin structures are respectively associated with

$$A:(z,\theta)\mapsto(z+1,-\theta)\qquad B:(z,\theta)\mapsto(z+\tau,\theta)$$

and

$$A:(z,\theta)\mapsto(z+1,-\theta)\qquad B:(z,\theta)\mapsto(z+\tau,-\theta)\,.$$

We recall the fact that the $SL_2(\mathbb{Z})$ action preserves the odd spin structure on X_τ and the modular transformation

$$(z,\tau)\mapsto\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\qquad\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$$

transforms the even spin structures into one another. The subgroups of $SL_2(\mathbb{Z})$ which preserve an even spin structure are conjugate to

$$\Gamma_0(2):=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z}):\begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}1&\star\\0&1\end{pmatrix}\bmod 2\right\}.\tag{13}$$

Let $\hat{\Gamma}_0(2)$ be the lift of $\Gamma_0(2)$ to the metaplectic group $Mp_2(\mathbb{R})$. The necessity of introducing $Mp_2(\mathbb{R})$ is just a way to allow taking square roots of functions!

Let $\Upsilon_{\mathrm{ev}}\to H$, where H is the upper half plane, be the family of even supertori $\mathbb{C}^{1|1}\times_{\mathbb{Z}^2}H\to H$.

Since an even family $\Upsilon_{\mathrm{ev}}\to H$ is given by a quotient of $\mathbb{C}^{1|1}$ with respect to two generators, we will have four super theta functions after Levin [3].

Theorem 5. The series

$$\sum_{m\in\mathbb{Z}}\exp\left[2\pi\mathrm{i}\left(\frac{(m+\delta)^2}{2}\tau+(m+\delta)(z+\varepsilon)\right),(-1)^{m+\delta}\zeta\right]\tag{14}$$

where $\exp[u,\xi]:=(\exp u)(1+\xi)=\exp(u+\xi)$ is the exponential function on $\mathbb{C}^{1|1}$, $\delta,\varepsilon\in\{0,\frac{1}{2}\}$, converges absolutely and uniformly on compacta and thus defines a holomorphic function $\vartheta^{\mathrm{ev}}_{2\delta,2\varepsilon}(z,\zeta\mid\tau)$ on $\mathbb{C}^{1|1}\times H$, called the even super theta function with characteristics δ and ε.

We want to relate the even super theta functions to the super Schrödinger representation. We take the Gaussian $v_\tau(x)=\exp\pi\mathrm{i}\tau x^2$ with $\tau\in H$ where (x,θ) is the coordinate on $\mathbb{R}^{1|1}$, then v_τ is holomorphic function of τ and as a function of (x,θ) it belongs to $S(\mathbb{R}^{1|1})$. We compute for $k,l\mid\xi,\eta)\in\mathbb{R}^{2|2}$,

$$\mu(\exp ka)\mu(\exp la^+)\mu(\exp\sigma\alpha)\mu(\exp\rho\alpha^+)v_\tau\tag{15}$$

$$=\exp[2\pi\mathrm{i}kx+\pi\mathrm{i}\tau(x+l)^2]\exp 2\pi\mathrm{i}\sigma\vartheta.\tag{16}$$

Theorem 6. We define the 'theta distribution' λ on $L^2(\mathbb{R}^{1|1})$ so that

$$\langle \lambda, f(x)+g(x)\vartheta \rangle = \sum_{m\in\mathbb{Z}} f(m+\delta)+(-1)^{m+\delta}g(m+\delta) \tag{17}$$

where $\delta \in \{0, \frac{1}{2}\}$ and $f(x)+g(x)\vartheta \in L^2(\mathbb{R}^{1|1})$. Then we have

$$\vartheta^{\mathrm{ev}}_{2\delta,0}(z,\zeta\,|\,\tau)=\langle \lambda, \mu(\exp ka)\mu(\exp la^+)\mu(\exp \sigma\alpha)\mu(\exp \rho\alpha^+)v_\tau \rangle. \tag{18}$$

This is holomorphically dependent on z and σ, which are the appropriate holomorphic coordinates on $\mathbb{C}^{1|1}$ with the odd 'imaginary' part ρ degenerated.

Proof. We have

$$\langle \lambda, \mu(\exp ka)\mu(\exp la^+)\mu(\exp \sigma\alpha)\mu(\exp \rho\alpha^+)v_\tau \rangle$$

$$= \sum_{m\in\mathbb{Z}} \exp 2\pi i\left[\frac{(m+\delta+l)^2}{2}\tau+(m+\delta)k+(-1)^{m+\delta}(m+\delta)\sigma\right]$$

$$= \sum_{m\in\mathbb{Z}} \exp 2\pi i\left[\frac{(m+\delta)^2}{2}\tau+(m+\delta)z+(-1)^{m+\delta}(m+\delta)\sigma\right] \tag{19}$$

where $z:=k+l\tau$. It is clear that we obtain $\vartheta^{\mathrm{ev}}_{2\delta,0}(z,\zeta\,|\,\tau)$ as the matrix coefficient of $sH(2|2)$.

In conclusion, we do not get a representation of the semi-direct product $sH(2|2)$ with $OSP(2|2)$ here, nor have we described explicitly what the OSP action is, even though the infinitesimal action has been given above. We would like to address these issues elsewhere. However, what we have done so far suggests a more geometric approach to super theta functions and the Weil representation even though there are important differences between the even and odd cases because the existence of a cubic term m^3 in the definition of odd super theta functions [17] means that we cannot work with the Gaussian which can only provide a quadratic term m^2 as in the definition of the even super theta functions.

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